Disjunctive Syllogism in Relevant Logics

@ Bergen Workshop on Logical Disagreements

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Basic idea of relevant logics

- Relevant logics:
  - Some conditional $\rightarrow$ *expresses* entailment
  - An argument can't be valid if the premises are unrelated to the conclusion.
  - A $\rightarrow$-statement can't be logically true if the antecedent is unrelated to the consequent.
- Paradoxes of the material and strict conditionals:
- Classical logic: $A \wedge \sim A \supset B$
- Modal logic: $A \wedge \sim A \dashv B$
- Neither $\supset$ nor $\dashv$ can express entailment

Traditional claim: relevant logics are all paraconsistent

- (Berto & Restall)

  > *Relevant logic is paraconsistent, so we all count as paraconsistentists to the extent that we count as relevantists. ([BR19, p. 23])*

- (Read)

  > *Recently Bob Meyer has claimed that relevant logic is mistaken in rejecting* $DS_\vee$*[. . .]. In contrast I claim that the rejection of* $DS_\vee$ *is central to the whole conception of relevant logic. ([Rea81, p. 66])*

  $DS_\vee =_{df}$ *the inference schema* **if** A **and** $\ulcorner \sim A \vee B \urcorner$ **are true, then so is** B

Cont.: Traditional claim: relevant logics are all paraconsistent

- (Burgess)

  All relevantists agree in rejecting disjunctive syllogism (DS):

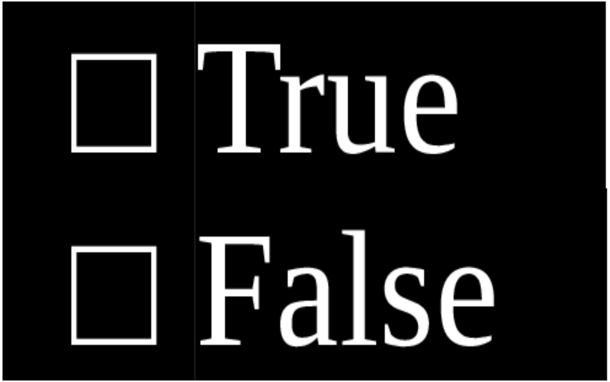
$$(DS) \qquad \frac{\begin{array}{l} p \vee q \\ {\sim}p \end{array}}{q}$$

  ([Bur83, p. 41])

- (Anderson & Belnap)

  we do hold that the inference from $\overline{A}$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make [...]. Such an inference commits nothing less than a fallacy of relevance. ([AB75, p. 165])

Relevance $\Rightarrow$ Paraconsistency?



Plan of the talk

- Definitions, such as explosive vs. paraconsistent logic
- Show forth Ackermann's logic Π' and Anderson and Belnap's **E**
- Show forth an explosive logic which has all the treasured properties Anderson and Belnap's **E**
- Time permitting: look at the concept of relevant deduction

Disjunctive syllogism, Explosion & Paraconsistency

Definition (Explosion)

A consequence relation $\vdash$ is *explosive* just in case for all A's and B's,

$$\{A, \sim A\} \vdash B.$$

Definition (Paraconsistency)

A consequence relation $\vdash$ is *paraconsistent* just in case explosion does not hold for it.

Definition (Disjunctive syllogism)

A consequence relation $\vdash$ validates *disjunctive syllogism* just in case for all A's and B's,

$$\{A, \sim A \vee B\} \vdash B$$

The Hilbert consequence relation of a logic

Definition ($\vdash^h$)

A HILBERT PROOF of a formula A from a set of formulas Γ in the logic **L** is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of **L**, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of **L**. The existential claim that there is such a proof is written $\Gamma \vdash^h_{\mathbf{L}} A$ and expressed as "there exists a Hilbert-derivation of A from Γ in the logic **L**".

$$A_1, A_2, A_3, \ldots, , A_n$$

- $A_1 \in \Gamma$
- $A_1 \in AXIOM_{\mathbf{L}}$

Relevance properties

Definition (Variable sharing)

A logic $\mathbf{L}$ has the *variable sharing property* $=_{df}$
if $\vdash^h_{\mathbf{L}} A \to B$, then A and B share a propositional variable.

Definition (Motivational: Relevant deduction property / Entailment Theorem)

A logic has the *relevant deduction property* just in case $A \to B$ is a logical theorem if (and only if) there is a proof of B in which A is *used*

Definition (Anderson & Belnap: Relevant logic)

- Necessary property: the variable sharing property
- Necessary and sufficient property: the relevant deduction property / Entailment theorem

Anderson and Belnap's startingpoint: Ackermann's Π' ([Ack56])

Definition (Ackermann's Π')

(Ax1) $A \to A$
(Ax2) $A \to A \vee B$ and $B \to A \vee B$
(Ax3) $A \wedge B \to A$ and $A \wedge B \to B$
(Ax4) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(Ax6) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(Ax7) $(A \to (A \to B)) \to (A \to B)$
(Ax8) $(A \to B) \to ((C \to A) \to (C \to B))$
(Ax9) $(A \to B) \to ((B \to C) \to (A \to C))$

(Ax10) $\sim\sim A \to A$
(Ax11) $(A \to \sim B) \to (B \to \sim A)$
(Ax12) $(A \to \sim A) \to \sim A$

(α) $\{A, A \to B\} \Vdash B$
(β) $\{A, B\} \Vdash A \wedge B$
(γ) $\{A, \sim A \vee B\} \Vdash B$
(δ) $\{A \to (B \to C), B\} \Vdash A \to C$

Π' is $\vdash^h$-explosive

Theorem

Π' is $\vdash^h$-explosive, that is $\{A, \sim A\} \vdash^h_{\Pi'} B$ for all A's and B's.

Proof.

(1)	A	*assumption*
(2)	$\sim A$	*assumption*
(3)	$\sim A \to \sim A \vee B$	*(Ax2)*
(4)	$\sim A \vee B$	*2, 3,* (α)
(5)	B	*1, 4,* (γ)

□

Anderson and Belnap's Π'-verdict

- Pro: Seems to block the paradoxes of the material and strict conditionals
- Pro: Can express **S4**-modality using a defined $\Box$
- Con: Lacks a deduction theorem for $\rightarrow$
- Solution: drop (γ) and (δ).

$$
\begin{array}{ll}
(\gamma) & \{A, {\sim}A \lor B\} \Vdash B \\
(\delta) & \{A \rightarrow (B \rightarrow C), B\} \Vdash A \rightarrow C
\end{array}
$$

Anderson and Belnap's **E**

Definition (Anderson and Belnap's **E**)

Anderson and Belnap's **E** consists of axioms (Ax1)–(Ax12), together with the rules (α) and (β) above, as well as the following axioms:

$$
\begin{array}{ll}
(\text{Ax13}) & ((A \to A) \to B) \to B \\
(\text{Ax14}) & \Box A \land \Box B \to \Box(A \land B) \quad \Box C =_{df} (C \to C) \to C
\end{array}
$$

Fact ((γ) and (δ) are admissible in **E**)

$$
\begin{array}{lll}
(\gamma^a) & \varnothing \vdash^h_{\mathbf{E}} A \ \& \ \varnothing \vdash^h_{\mathbf{E}} {\sim}A \lor B \Longrightarrow \varnothing \vdash^h_{\mathbf{E}} B & \text{[MD69]} \\
(\delta^a) & \varnothing \vdash^h_{\mathbf{E}} A \to (B \to C) \ \& \ \varnothing \vdash^h_{\mathbf{E}} B \Longrightarrow \varnothing \vdash^h_{\mathbf{E}} A \to C & \text{[AB75, §8.2]}
\end{array}
$$

Corollary

$$\varnothing \vdash^h_{\mathbf{E}} A \Longleftrightarrow \varnothing \vdash^h_{\mathbf{\Pi'}} A$$

Fact

E *is* $\vdash^h$*-paraconsistent*

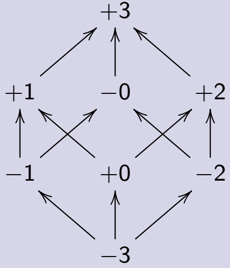
Belnap: ([Bel60]) Π' and **E** have the variable sharing property

Theorem

If $\vdash^h_{\mathbf{L}} A \to B$ *for* $\mathbf{L} \in \{\Pi', \mathbf{E}\}$*, then* A *and* B *share a propositional variable.*

Proof.

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\sim$	$\Box$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	-3
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$	$+2$	-2
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$	$+1$	-1
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$	$+0$	-0
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$	-2	$+2$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$	-3	$+3$

Figure: *Belnap's model of relevance*

More Π'- and **E**-facts

Fact ([AB59]: Π' and E are S4-type modal logics)

For **L** $\in \{\Pi', \mathbf{E}\}$,

$$
\begin{array}{rl}
(\mathbf{K}) & \vdash^h_{\mathbf{L}} \Box(A \to B) \to (\Box A \to \Box B) \\
(\mathbf{T}) & \vdash^h_{\mathbf{L}} \Box A \to A \\
(\mathbf{4}) & \vdash^h_{\mathbf{L}} \Box A \to \Box\Box A \\
(\mathbf{N}) & \varnothing \vdash^h_{\mathbf{L}} A \Longrightarrow \varnothing \vdash^h_{\mathbf{L}} \Box A
\end{array}
$$

In fact, the search for a suitable deduction theorem for Ackermann's systems [...] provided the initial impetus leading to the research reported in this book. ([AB75, p. 261])

Fact (E-Enthymematical deduction thm.)

For every set of formulas Θ, there is a conjunction C of axioms of **E** *such that*

$$\Theta \vdash^h_{\mathbf{E}} A \Longleftrightarrow \varnothing \vdash^h_{\mathbf{E}} \bigwedge_{i \le n} \theta_i \wedge C \to A$$

Cont.: More Π'- and **E**-facts

Fact (The enthymematical deduction theorem fails for Π')

Proof.

$$A, \sim A \vee B \vdash^{h}_{\mathbf{\Pi'}} B,$$

but

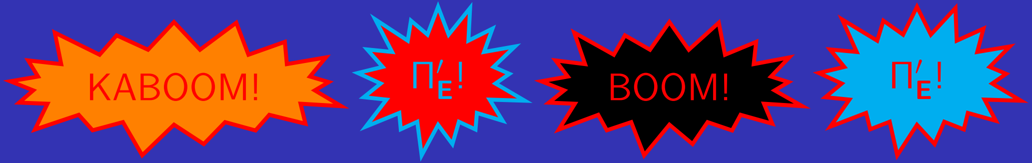
$$\varnothing \nvdash^{h}_{\mathbf{\Pi'}} (A \wedge (\sim A \vee B)) \wedge C \to B$$

(ref. [AB75, §25.1]) □

Suppose now for the sake of argument that The Dog reasons as follows: "The arguments of §§16 and 22 make it clear that, when The Man accepts $A \& (\overline{A} \vee B) \rightarrow B$, he is making a simple inferential blunder. But surely The Man has something in mind [...].
Maybe The Man meant that there were some axioms Axioms which when conjoined to the premises, would produce the desired entailment: he may have supposed that $\vdash A \& (\overline{A} \vee B) \& Axioms \rightarrow B$. This could have happened if, being as confused as he is, he was thinking of the Official deduction theorem [...] [A&B notes that any such formula can be falsified in Belnap's test-model]. So B does not even derive "from" A and $\overline{A} \vee B$ in the Official sense [...]. [AB75, p. 297f.]

Desiderata for relevant $\vdash^h$-explosion—combining **E** and Π'

- Retain **E** and Π''s variable sharing property
- Retain **E** and Π''s **S4**-modality
- Retain **E**'s enthymematical deduction theorem
- Retain Π''s derivability of disjunctive syllogism / explosion
- Retain **E**'s Entailment theorem



Definition (Π'_E)

(Ax1) $A \to A$

(Ax2) $A \to A \vee B$ and $B \to A \vee B$

(Ax3) $A \wedge B \to A$ and $A \wedge B \to B$

(Ax4) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

(Ax5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(Ax6) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(Ax7) $(A \to (A \to B)) \to (A \to B)$

(Ax8$^\flat$) $(A \to B) \wedge (C \to C) \to ((C \to A) \to (C \to B))$

(Ax9$^\flat$) $(A \to B) \wedge (C \to C) \to ((B \to C) \to (A \to C))$

(Ax10) $\sim\sim A \to A$

(Ax11) $(A \to \sim B) \to (B \to \sim A)$

(Ax12) $(A \to \sim A) \to \sim A$

(Ax13) $((A \to A) \to B) \to B$

(Ax14) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(**K**) $\Box(A \to B) \to (\Box A \to \Box B)$

(**4**) $\Box A \to \Box\Box A$

(Ax15) $(A \to B) \wedge (B \to C) \to (A \to C)$

(Ax16) $(A \wedge (\sim A \vee B)) \wedge (B \to B) \to B$

(α) $\{A, A \to B\} \Vdash B$

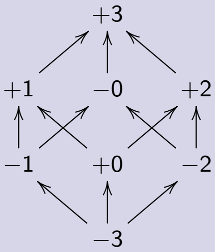
(β) $\{A, B\} \Vdash A \wedge B$

Theorem

$\Pi'_{\mathbf{E}}$ has the variables sharing property.

Proof.

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\sim$	$\Box$
-3	$+0$	$+2$	$+1$	$+3$	$+0$	$+1$	$+2$	$+3$	$+3$	-3
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+2$	$+2$	-2
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+1$	$+1$	-1
-0	-3	-3	-3	$+0$	-3	-3	-3	$+0$	$+0$	-0
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+1$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+2$	-2	$+2$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+0$	-3	$+3$

Figure: $\Pi'_{\mathbf{E}}$*'s model of relevance*

$\Pi'_{\mathbf{E}}$: **S4**-modality and the enthymematical deduction theorem

Theorem

$\Pi'_{\mathbf{E}}$'s $\Box$ has the characteristics of a **S4** *modality*

Proof.

The **T***-axiom: instance of (Ax13); the others are primitive axioms of $\Pi'_{\mathbf{E}}$. The admissibility of the necessitation rule is a simple induction on the length of proof.* $\Box$

Theorem

$\Pi'_{\mathbf{E}}$ has enthymematical deduction theorem.

Proof.

Simple induction on length of proof. $\Box$

$\vdash^{h}_{\Pi'_E}$ is explosive

Theorem

For all A's and B's, $\{A, {\sim}A\} \vdash^{h}_{\Pi'_E} B$

Proof.

(1)	A	*assumption*
(2)	${\sim}A$	*assumption*
(3)	${\sim}A \to {\sim}A \vee B$	*(Ax2)*
(4)	${\sim}A \vee B$	*2, 3,* (α)
(5)	$A \wedge ({\sim}A \vee B)$	*1, 4,* (β)
(6)	$B \to B$	*(Ax1)*
(7)	$(A \wedge ({\sim}A \vee B)) \wedge (B \to B)$	*5, 6,* (β)
(8)	$(A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B$	*(Ax16)*
(9)	B	*7, 8,* (α)

□

The relevant consequence relation of a logic

Definition (Relevant deduction)

$\Gamma \vdash^r_{\mathbf{L}} A$: A RELEVANT DEDUCTION of a formula A from a set of formulas Γ in the logic **L** having only modus ponens, (α), and adjunction, (β), as primitive rules, is defined as a Hilbert proof $A_1, \ldots, A_n$ of A from Γ such that it is possible to mark the A_i's with $\#$'s according to the following rules:

1. If $A_i \in \Gamma$, then A_i is marked.
2. If A_i is got from A_j and A_k using modus ponens, then A_i is marked if either or both of A_j and A_k are marked.
3. Adjunction is only used on premises which are either both marked or both unmarked.
4. If A_i is got from A_j and A_k using adjunction and both of A_j and A_k are marked, then A_i is marked.
5. No other formulas are marked.
6. As a consequence of (1–5), A_n is marked.

How to think of the relevant consequence relation of a logic

$$\Theta \vdash^{r}_{\mathbf{L}} A \quad =_{df} \quad \vdash^{h}_{\mathbf{L}} \bigwedge_{i \leq n} \theta_i \rightarrow A$$

for some subset $\{\theta_1, \ldots, \theta_n\} \subseteq \Theta$.

Cont.: $\vdash^h_{\Pi'_E}$ vs. $\vdash^r_{\Pi'_E}$

Theorem

$\vdash^r_{\Pi'_E}$ *is paraconsistent: for some A's and B's,* $\{A, {\sim}A\} \nvdash^r_{\Pi'_E} B$.

Failed Proof

(1)	#	A	assumption
(2)	#	${\sim}A$	assumption
(3)		${\sim}A \rightarrow {\sim}A \vee B$	(Ax2)
(4)	#	${\sim}A \vee B$	2, 3, (α)
(5)	#	$A \wedge ({\sim}A \vee B)$	1, 4, (β)
(6)		$B \rightarrow B$	(Ax1)
(7)		$(A \wedge ({\sim}A \vee B)) \wedge (B \rightarrow B)$	5, 6, (β): Not allowed: 3.clause on use of (β)
(8)		$(A \wedge ({\sim}A \vee B)) \wedge (B \rightarrow B) \rightarrow B$	(Ax16)
(9)		B	7, 8, (α); not marked

Theorem ($\vdash^{r}_{\Pi'_E}$ is only irregularly paraconsistent)

$\{A, \sim A, B \rightarrow B\} \vdash^{r}_{\Pi'_E} B$

Proof.

(1)	#	A	*assumption*
(2)	#	$\sim A$	*assumption*
(3)		$\sim A \rightarrow \sim A \vee B$	*(Ax2)*
(4)	#	$\sim A \vee B$	*2, 3, (α)*
(5)	#	$A \wedge (\sim A \vee B)$	*1, 4, (β)*
(6)	#	$B \rightarrow B$	*assumption*
(7)	#	$(A \wedge (\sim A \vee B)) \wedge (B \rightarrow B)$	*5, 6, (β)*
(8)		$(A \wedge (\sim A \vee B)) \wedge (B \rightarrow B) \rightarrow B$	*(Ax16)*
(9)	#	B	*7, 8, (α)*

□

The Entailment theorem

Theorem (The Entailment theorem)

Let **L** *be any axiomatic extension of either* **E** *or* $\Pi'_{\mathbf{E}}$. *Then*

$$\{\psi_1, \ldots, \psi_n\} \cup \{\theta_1, \ldots, \theta_m\} \vdash^r_{\mathbf{L}} B \iff \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \leq n} \psi_i \wedge \bigwedge_{i \leq m} \theta_i \to B$$

Where $\{\theta_1, \ldots, \theta_m\} \subseteq \mathit{Axioms}_{\mathbf{L}}$.

Proof.

Easy induction on the length of proof. □

Corollary (Entailment thm. combined with the enthymematical ded. theorem)

Let **L**, *the* θ*'s and* ψ*'s be as above. Then*

$$\{\psi_1, \ldots, \psi_n\} \cup \{\theta_1, \ldots, \theta_m\} \vdash^r_{\mathbf{L}} B \iff \{\psi_1, \ldots, \psi_n\} \vdash^h_{\mathbf{L}} B$$

Cont.: The Entailment theorem

Corollary

The Entailment theorem holds for classical logic...

Bad Definition (The definition of 'relevant logic' is bad)

Classical logic does not have the variable sharing property—a necessary property of being a relevant logic—yet satisfies the Entailment theorem—a necessary and *sufficient* for being a relevant logic.

Cont.: $\vdash^h_{\Pi'_\mathbf{E}}$ vs. $\vdash^r_{\Pi'_\mathbf{E}}$

Proof that $\vdash^r_{\Pi'_\mathbf{E}}$ is paraconsistent.

Given the Entailment Theorem:

$$\{A, \sim A\} \vdash^r_{\Pi'_\mathbf{E}} B \Longleftrightarrow \varnothing \vdash^r_{\Pi'_\mathbf{E}} A \wedge \sim A \rightarrow B.$$

Since $A \wedge \sim A \rightarrow B$ fails in $\Pi'_\mathbf{E}$'s model of relevance, it follows that $\{A, \sim A\} \nvdash^r_{\Pi'_\mathbf{E}} B$. □

Corollary

If **L** *is a logic for which the Entailment theorem holds, then* $\vdash^r_\mathbf{L}$ *is paraconsistent if* **L** *has the variable sharing property.*

Summing up

We have seen that $\Pi'_{\mathbf{E}}$ has the following properties

- Variable sharing
- Entailment theorem
- Enthymematical deduction theorem
- **S4** modality
- Explosive $\vdash^h$
- Mildly paraconsistent $\vdash^r$

Cont.: Summing up

we do hold that the inference from $\overline{A}$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make [...]. Such an inference commits nothing less than a fallacy of relevance. ([AB75, p. 165])

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