Non-Boolean Classical Relevant Logics

@ LanCog Workshop on Substructural Logics, Lisboa

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Basic idea of relevant logics

- Some of the motivation behind relevant logic:
  - Some conditional $\rightarrow$ *expresses* entailment
  - An argument can't be valid if the premises are unrelated to the conclusion.
    - A $\rightarrow$-statement can't be logically true if the antecedent is unrelated to the consequent.
    - It can't be the case that B is a logical consequence of A if there is a proof of B which does not make use of A.
- Implicational paradoxes of the material and strict conditionals:
  - Classical logic: $A \wedge {\sim}A \supset B$
  - Modal logic: $A \wedge {\sim}A \prec B$

Relevant dictum:

- Neither $\supset$ nor $\prec$ can express entailment
- Disjunctive Syllogism is invalid

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Traditional claim: relevant logics are all paraconsistent

- (Berto & Restall)

  Relevant logic is paraconsistent, so we all count as paraconsistentists to the extent that we count as relevantists. ([BR19, p. 23])

- (Read)

  Recently Bob Meyer has claimed that relevant logic is mistaken in rejecting $DS_\vee$ [. . .]. In contrast I claim that the rejection of $DS_\vee$ is central to the whole conception of relevant logic. ([Rea81, p. 66])

  $DS_\vee =_{df}$ *the inference schema* **if** A **and** $\ulcorner{\sim}A \vee B\urcorner$ **are true, then so is** B

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- (Burgess)

  All relevantists agree in rejecting disjunctive syllogism (DS):

$$(DS) \quad \frac{\begin{array}{l} p \vee q \\ \sim p \end{array}}{q}$$

  ([Bur83, p. 41])

- (Anderson & Belnap)

  we do hold that the inference from $\overline{A}$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make [...]. Such an inference commits nothing less than a fallacy of relevance. ([AB75, p. 165])

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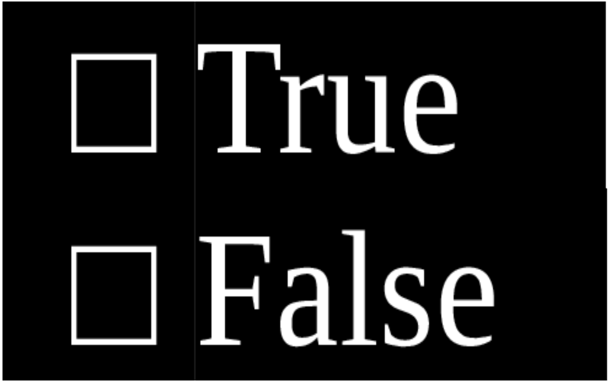
The purpose of this talk

Relevance $\Rightarrow$ Paraconsistency?



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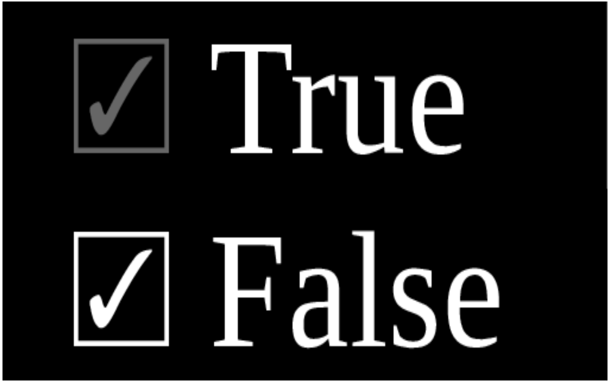

What is Relevance?

Plan of the talk

- Definitions, such as explosive vs. paraconsistent consequence relations.
- Show forth Ackermann's logic Π' and Anderson and Belnap's **E**.
- Show forth a Hilbert-explosive logic which has all the treasured properties of Anderson and Belnap's **E** and **R**.
- Show that virtually any logic with no more rules than modus ponens and adjunction satisfies the Entailment Theorem (including classical logic).
- Show how to extends logics like **E** and **R** into the Hilbert-explosive logics **Æ** and **M** which can represent both their Hilbert- and relevant consequence relations. (spoiler: got by adding $\mathbf{t} \wedge \mathbf{f} \to A$)
- Say something about the intra-theoretical pluralism inherent in **Æ** and **M**

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Disjunctive syllogism, Explosion & Paraconsistency

Definition (Explosion)

A consequence relation $\vdash$ is *explosive* just in case for all A's and B's,

$$\{A, \sim A\} \vdash B.$$

Definition (Paraconsistency)

A consequence relation $\vdash$ is *paraconsistent* just in case explosion does not hold for it.

Definition (Disjunctive syllogism)

A consequence relation $\vdash$ validates *disjunctive syllogism* just in case for all A's and B's,

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The Hilbert consequence relation of a logic

Definition ($\vdash^h$)

A HILBERT PROOF of a formula A from a set of formulas Γ in the logic **L** is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of **L**, or there is a set $\Delta \subseteq \{A_j \mid j < i\}$ such that $\Delta \Vdash A_i$ is an instance of a rule of **L**. The existential claim that there is such a proof is written $\Gamma \vdash^h_{\mathbf{L}} A$ and expressed as "there exists a Hilbert-derivation of A from Γ in the logic **L**".

$$A_1, A_2, A_3, \ldots, A_n$$

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- $A_1 \in AXIOM_{\mathbf{L}}$

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Relevance properties

Definition (Variable sharing)

A logic $\mathbf{L}$ has the *variable sharing property* $=_{df}$
if $\vdash^{h}_{\mathbf{L}} A \rightarrow B$, then A and B share a propositional variable.

Definition (Relevant deduction property / Entailment Theorem)

A logic has the *relevant deduction property* just in case $A \rightarrow B$ is a logical theorem if (and only if) there is a proof of B in which A is *used*

Definition (Anderson & Belnap: Relevant logic)

- Necessary property: the variable sharing property
- Necessary and sufficient property: the relevant deduction property / Entailment theorem

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Anderson and Belnap's startingpoint: Ackermann's Π' ([Ack56])

Definition (Ackermann's Π')

(Ax1) $A \to A$
(Ax2) $A \to A \vee B$ and $B \to A \vee B$
(Ax3) $A \wedge B \to A$ and $A \wedge B \to B$
(Ax4) $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax5) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(Ax6) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(Ax7) $(A \to (A \to B)) \to (A \to B)$
(Ax8) $(A \to B) \to ((C \to A) \to (C \to B))$
(Ax9) $(A \to B) \to ((B \to C) \to (A \to C))$

(Ax10) $\sim\sim A \to A$
(Ax11) $(A \to \sim B) \to (B \to \sim A)$
(Ax12) $(A \to \sim A) \to \sim A$

(α) $\{A, A \to B\} \Vdash B$
(β) $\{A, B\} \Vdash A \wedge B$
(γ) $\{A, \sim A \vee B\} \Vdash B$
(δ) $\{A \to (B \to C), B\} \Vdash A \to C$

Π' is $\vdash^h$-explosive

Theorem

Π' is $\vdash^h$-explosive, that is $\{A, {\sim}A\} \vdash^h_{\Pi'} B$ for all A's and B's.

Proof.

(1)	A	*assumption*
(2)	${\sim}A$	*assumption*
(3)	${\sim}A \rightarrow {\sim}A \vee B$	*(Ax2)*
(4)	${\sim}A \vee B$	*2, 3,* (α)
(5)	B	*1, 4,* (γ)

□

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(1)	A	*assumption*
(2)	$\sim A$	*assumption*
(3)	$\sim A \to \sim A \vee B$	*(Ax2)*
(4)	$\sim A \vee B$	*2, 3, (α)*
(5)	B	*1, 4, (γ)*

□

Anderson and Belnap's Π'-verdict

- Pro: Seems to block the paradoxes of the material and strict conditionals
- Pro: Can express **S4**-modality using a defined $\Box$
- Con: Lacks a deduction theorem for $\to$
- Solution: drop (γ) and (δ).

$$(\gamma) \quad \{A, {\sim}A \vee B\} \Vdash B$$
$$(\delta) \quad \{A \to (B \to C), B\} \Vdash A \to C$$

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\end{array}
$$

Anderson and Belnap's Π'-verdict

- Pro: Seems to block the paradoxes of the material and strict conditionals
- Pro: Can express **S4**-modality using a defined $\Box$
- Con: Lacks a deduction theorem for $\rightarrow$
- Solution: drop (γ) and (δ).

$$
\begin{array}{ll}
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(\delta) & \{A \rightarrow (B \rightarrow C), B\} \Vdash A \rightarrow C
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Anderson and Belnap's **E**

Definition (Anderson and Belnap's **E**)

Anderson and Belnap's **E** consists of axioms (Ax1)–(Ax12), together with the rules (α) and (β) above, as well as the following axioms:

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\begin{array}{ll}
\text{(Ax13)} & ((A \rightarrow A) \rightarrow B) \rightarrow B \\
\text{(Ax14)} & \Box A \wedge \Box B \rightarrow \Box(A \wedge B) \quad \Box C =_{df} (C \rightarrow C) \rightarrow C
\end{array}
$$

Fact ((γ) and (δ) are admissible in **E**)

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\begin{array}{lll}
(\gamma^a) & \varnothing \vdash^h_{\mathbf{E}} A \;\&\; \varnothing \vdash^h_{\mathbf{E}} {\sim}A \vee B \Longrightarrow \varnothing \vdash^h_{\mathbf{E}} B & \text{[MD69]} \\
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\end{array}
$$

Corollary

$$\varnothing \vdash^h_{\mathbf{E}} A \Longleftrightarrow \varnothing \vdash^h_{\mathbf{\Pi'}} A$$

Fact

E *is* $\vdash^h$*-paraconsistent*

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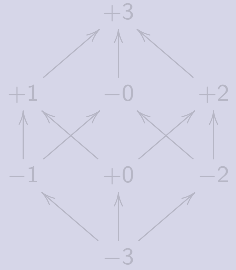
Theorem

If $\vdash^h_{\mathbf{L}} A \to B$ *for* $\mathbf{L} \in \{\Pi', \mathbf{E}\}$, *then* A *and* B *share a propositional variable.*

Proof.

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\sim$	$\Box$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	-3
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$	$+2$	-2
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$	$+1$	-1
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$	$+0$	-0
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$	-2	$+2$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$	-3	$+3$

Figure: *Belnap's model of relevance (note: $+0 \wedge -0 = -3$)*

Belnap: ([Bel60]) Π' and **E** have the variable sharing property

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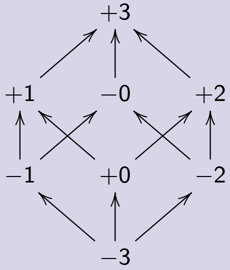

$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\sim$	$\Box$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	-3
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$	$+2$	-2
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$	$+1$	-1
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$	$+0$	-0
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$	-2	$+2$
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More Π'- and **E**-facts

Fact ([AB59]: Π' and E are S4-type modal logics)

For $\mathbf{L} \in \{\Pi', \mathbf{E}\}$,

$$\begin{array}{rl} (\mathbf{K}) & \varnothing \vdash^h_{\mathbf{L}} \Box(A \to B) \to (\Box A \to \Box B) \\ (\mathbf{T}) & \varnothing \vdash^h_{\mathbf{L}} \Box A \to A \\ (\mathbf{4}) & \varnothing \vdash^h_{\mathbf{L}} \Box A \to \Box\Box A \\ (\mathbf{N}) & \varnothing \vdash^h_{\mathbf{L}} A \Longrightarrow \varnothing \vdash^h_{\mathbf{L}} \Box A \end{array}$$

In fact, the search for a suitable deduction theorem for Ackermann's systems [...] provided the initial impetus leading to the research reported in this book. ([AB75, p. 261])

Fact (E-Enthymematical deduction thm.)

For every set of formulas Θ, there is a conjunction C of axioms of **E** *such that*

$$\Theta \vdash^h_{\mathbf{E}} A \Longleftrightarrow \varnothing \vdash^h_{\mathbf{E}} \bigwedge_{i \leq n} \theta_i \wedge C \to A$$

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(\mathbf{T}) & \varnothing \vdash^h_{\mathbf{L}} \Box A \to A \\
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Cont.: More Π'- and **E**-facts

Fact (The enthymematical deduction theorem fails for Π')

Proof.

$$A, \sim A \vee B \vdash^h_{\Pi'} B,$$

but

$$\varnothing \nvdash^h_{\Pi'} (A \wedge (\sim A \vee B)) \wedge C \to B$$

(ref. [AB75, §25.1]) □

Corollary

It is not the case that for every choice of A's and B's that there exists an axiomatic conjunction C such that

$$\varnothing \nvdash^h_{\mathbf{E}[\gamma]} (A \wedge (\sim A \vee B)) \wedge C \to B \quad \varnothing \nvdash^h_{\mathbf{R}[\gamma]} (A \wedge (\sim A \vee B)) \wedge C \to B$$

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The confused Man

Suppose now for the sake of argument that The Dog reasons as follows: "The arguments of §§16 and 22 make it clear that, when The Man accepts $A\&(\overline{A} \vee B) \to B$, he is making a simple inferential blunder. But surely The Man has something in mind [...].
Maybe The Man meant that there were some axioms Axioms which when conjoined to the premises, would produce the desired entailment: he may have supposed that $\vdash A\&(\overline{A} \vee B)\&Axioms \to B$. This could have happened if, being as confused as he is, he was thinking of the Official deduction theorem [...] [A&B notes that any such formula can be falsified in Belnap's test-model]. So B does not even derive "from" A and $\overline{A} \vee B$ in the Official sense [...]. [AB75, p. 297f.]

Desiderata for relevant $\vdash^h$-explosion—combining **E** and Π'

- Retain **E** and Π''s variable sharing property
- Retain **E** and Π''s **S4**-modality
- Retain **E**'s enthymematical deduction theorem
- Retain Π''s derivability of disjunctive syllogism / explosion
- Retain **E**'s Entailment theorem

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An explosive relevant logic

Definition ($\Pi'_{\mathbf{E}}$)

(Ax1)	$A \to A$
(Ax2)	$A \to A \vee B$ and $B \to A \vee B$
(Ax3)	$A \wedge B \to A$ and $A \wedge B \to B$
(Ax4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(Ax6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(Ax7)	$(A \to (A \to B)) \to (A \to B)$
(Ax8$^\flat$)	$(A \to B) \wedge (C \to C) \to ((C \to A) \to (C \to B))$
(Ax9$^\flat$)	$(A \to B) \wedge (C \to C) \to ((B \to C) \to (A \to C))$
(Ax10)	$\sim\sim A \to A$
(Ax11)	$(A \to \sim B) \to (B \to \sim A)$
(Ax12)	$(A \to \sim A) \to \sim A$
(Ax13)	$((A \to A) \to B) \to B$
(Ax14)	$\Box A \wedge \Box B \to \Box(A \wedge B)$
(**K**)	$\Box(A \to B) \to (\Box A \to \Box B)$
(**4**)	$\Box A \to \Box\Box A$
(Ax15)	$(A \to B) \wedge (B \to C) \to (A \to C)$
(Ax16)	$(A \wedge (\sim A \vee B)) \wedge (B \to B) \to B$
(α)	$\{A, A \to B\} \Vdash B$
(β)	$\{A, B\} \Vdash A \wedge B$

KABOOM! Π'_E! SKADOOSH! Π'_E!

Definition (Π'_E)

(Ax1)	$A \to A$
(Ax2)	$A \to A \vee B$ and $B \to A \vee B$
(Ax3)	$A \wedge B \to A$ and $A \wedge B \to B$
(Ax4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
(Ax5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
(Ax6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
(Ax7)	$(A \to (A \to B)) \to (A \to B)$
($\text{Ax8}^\flat$)	$(A \to B) \wedge (C \to C) \to ((C \to A) \to (C \to B))$
($\text{Ax9}^\flat$)	$(A \to B) \wedge (C \to C) \to ((B \to C) \to (A \to C))$
(Ax10)	${\sim}{\sim}A \to A$
(Ax11)	$(A \to {\sim}B) \to (B \to {\sim}A)$
(Ax12)	$(A \to {\sim}A) \to {\sim}A$
(Ax13)	$((A \to A) \to B) \to B$
(Ax14)	$\Box A \wedge \Box B \to \Box(A \wedge B)$
(**K**)	$\Box(A \to B) \to (\Box A \to \Box B)$
(**4**)	$\Box A \to \Box\Box A$
(Ax15)	$(A \to B) \wedge (B \to C) \to (A \to C)$
(Ax16)	$(A \wedge ({\sim}A \vee B)) \wedge (B \to B) \to B$
(α)	$\{A, A \to B\} \Vdash B$
(β)	$\{A, B\} \Vdash A \wedge B$

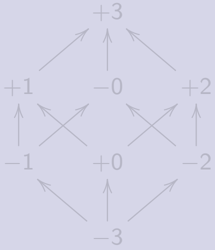
$\Pi'_{\mathbf{E}}$ and the the variable sharing property

Theorem

$\Pi'_{\mathbf{E}}$ *has the variable sharing property.*

Proof.

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\sim$	$\Box$
-3	$+0$	$+2$	$+1$	$+3$	$+0$	$+1$	$+2$	$+3$	$+3$	-3
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+2$	$+2$	-2
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+1$	$+1$	-1
-0	-3	-3	-3	$+0$	-3	-3	-3	$+0$	$+0$	-0
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+1$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+2$	-2	$+2$
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Figure: $\Pi'_{\mathbf{E}}$*'s model of relevance*

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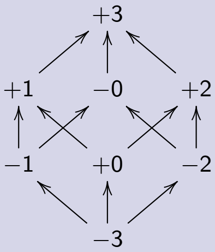

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-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+1$	$+1$	-1
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$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0	$+0$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+1$	-1	$+1$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+2$	-2	$+2$
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Figure: $\Pi'_{\mathbf{E}}$*'s model of relevance*

Π'_E: **S4**-modality and the enthymematical deduction theorem

Theorem

Π'_E's $\Box$ has the characteristics of a **S4** *modality*

Proof.

The **T***-axiom: instance of (Ax13); the others are primitive axioms of Π'_E. The admissibility of the necessitation rule is a simple induction on the length of proof.* $\Box$

Theorem

Π'_E has the enthymematical deduction theorem.

Proof.

Simple induction on length of proof. $\Box$

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$\Pi'_{\mathbf{E}}$ has the enthymematical deduction theorem.

Proof.

Simple induction on length of proof. $\Box$

$\vdash^h_{\Pi'_E}$ is explosive

Theorem

For all A's and B's, $\{A, \sim A\} \vdash^h_{\Pi'_E} B$

Proof.

(1)	A	*assumption*
(2)	$\sim A$	*assumption*
(3)	$\sim A \to \sim A \vee B$	*(Ax2)*
(4)	$\sim A \vee B$	*2, 3,* (α)
(5)	$A \wedge (\sim A \vee B)$	*1, 4,* (β)
(6)	$B \to B$	*(Ax1)*
(7)	$(A \wedge (\sim A \vee B)) \wedge (B \to B)$	*5, 6,* (β)
(8)	$(A \wedge (\sim A \vee B)) \wedge (B \to B) \to B$	*(Ax16)*
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(9)	B	*7, 8,* (α)

□

$\vdash^h_{\Pi'_E}$ extends classical logic

Theorem

Roughly: If τ translates any formula $A \to B$ into a new propositional variable, but leaves the other connectives unthouched, then if $\{\tau(A) | A \in \Delta\} \vdash^h_{\mathbf{TV}} \tau(B)$, then $\Delta \vdash^h_{\Pi'_E} B$.

Proof.

Induction on proof. □

Theorem (Material Deduction Theorem)

$$\Delta \cup \{A\} \vdash^h_{\Pi'_E} B \iff \Delta \vdash^h_{\Pi'_E} A \supset B$$

Proof.

Induction on proof. □

The relevant consequence relation of a logic

Definition (Relevant deduction)

$\Gamma \vdash^r_{\mathbf{L}} A$: A RELEVANT DEDUCTION of a formula A from a set of formulas Γ in the logic **L** having only modus ponens, (α), and adjunction, (β), as primitive rules, is defined as a Hilbert proof $A_1, \ldots, A_n$ of A from Γ such that it is possible to mark the A_i's with $\#$'s according to the following rules:

1. If $A_i \in \Gamma$, then A_i is marked.
2. If A_i is got from A_j and A_k using modus ponens, then A_i is marked if either or both of A_j and A_k are marked.
3. Adjunction is only used on premises which are either both marked or both unmarked.
4. If A_i is got from A_j and A_k using adjunction and both of A_j and A_k are marked, then A_i is marked.
5. No other formulas are marked.
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Why relevant deduction isn't substructral

- $\vdash^r$ isn't substructural:
  - Weakening: $\Gamma \vdash^r A \Rightarrow \Gamma \cup \Delta \vdash^r A$
  - Commutativity, associativity and contraction: immediate since $\vdash^r$ relates a *set* of formulas to a single formula.
  - Cut holds: if $\Gamma \vdash^r A$ and $\Delta \cup \{A\} \vdash^r B$, then $\Gamma \cup \Delta \vdash^r B$.
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A failed relevant proof

Theorem

$\vdash^r_{\Pi'_E}$ is paraconsistent: for some A's and B's, $\{A, \sim A\} \nvdash^r_{\Pi'_E} B$.

Failed Proof

(1)	#	A	assumption
(2)	#	$\sim A$	assumption
(3)		$\sim A \to \sim A \vee B$	(Ax2)
(4)	#	$\sim A \vee B$	2, 3, (α)
(5)	#	$A \wedge (\sim A \vee B)$	1, 4, (β)
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(7)		$(A \wedge (\sim A \vee B)) \wedge (B \to B)$	5, 6, (β): Not allowed: 3.clause on use of (β)
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Cont.: $\vdash^{h}_{\Pi'_{\mathbf{E}}}$ vs. $\vdash^{r}_{\Pi'_{\mathbf{E}}}$

Theorem ($\vdash^{r}_{\Pi'_{\mathbf{E}}}$ is only irregularly paraconsistent)

$\{A, \sim A, B \to B\} \vdash^{r}_{\Pi'_{\mathbf{E}}} B$

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The Entailment theorem

Theorem (The Entailment theorem)

Let **L** *be any axiomatic extension of either* **E** *or* $\Pi'_{\mathbf{E}}$. *Then*

$$\{\psi_1, \ldots, \psi_n\} \cup \{\theta_1, \ldots, \theta_m\} \vdash^r_{\mathbf{L}} B \iff \varnothing \vdash^h_{\mathbf{L}} \bigwedge_{i \le n} \psi_i \wedge \bigwedge_{i \le m} \theta_i \to B$$

Where $\{\theta_1, \ldots, \theta_m\} \subseteq \mathit{Axioms}_{\mathbf{L}}$.

Proof.

Easy induction on the length of proof. □

Corollary (Entailment thm. combined with the enthymematical ded. theorem)

Let **L**, *the* θ*'s and* ψ*'s be as above. Then*

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Cont.: The Entailment theorem

Corollary

The Entailment theorem holds for classical logic...

Bad Definition (The definition of 'relevant logic' is bad)

Classical logic does not have the variable sharing property—a necessary property of being a relevant logic—yet satisfies the Entailment theorem—a necessary and *sufficient* for being a relevant logic.

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Use and relevance

- Even though the Entailment Theorem also holds of classical logic, the consequence relation it captures is interesting as does give some content to notion of *use*
- "There is a Hilbert-derivation of $A \to A$ from $B \to B$" is correct, even though $B \to B$ is never used in the proof. This is simply because logical truths are *suppressible* for $\vdash^h$, to use Routley's term.
- Things are different for relevant derivability, however, since, $\emptyset \nvdash^r A$ for all A's.
- For logics with $(A \to A) \to (B \to B)$ as an axiom, one can get $B \to B$ by using $A \to A$.
- Heart of $\vdash^r$:
  - Logical and non-logical assumptions are treated equally!
  - Tempting to give an "anti-suppression reading" of it, but $\{A\} \vdash^r B \to A$ holds provided $A \to (B \to A)$ is an axiom...
  - Bottom line: Nothing to do with relevance!
- What makes $\vdash^r$ into a relation that has anything to do with relevance, is simply that the axioms chosen are such that the variable sharing property holds for the primary consequence relation $\vdash^h$.

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Relevance $\Longrightarrow$ paraconsistency?

Theorem ($\vdash^r_{\Pi'_{\mathbf{E}}}$ is paraconsistent)

Proof.

Given the Entailment Theorem:

$$\{A, {\sim}A\} \vdash^r_{\Pi'_{\mathbf{E}}} B \iff \varnothing \vdash^r_{\Pi'_{\mathbf{E}}} A \wedge {\sim}A \to B.$$

Since $A \wedge {\sim}A \to B$ *fails in* $\Pi'_{\mathbf{E}}$*'s model of relevance, it follows that* $\{A, {\sim}A\} \nvdash^r_{\Pi'_{\mathbf{E}}} B$. □

Corollary (Relevance $\Longrightarrow$ $\vdash^r$-paraconsistency)

If $\mathbf{L}$ *is a logic for which the Entailment theorem holds, then* $\vdash^r_{\mathbf{L}}$ *is paraconsistent if* $\mathbf{L}$ *has the variable sharing property.*

Relevance $\not\Longrightarrow$ $\vdash^h$-paraconsistency

Relevance $\Longrightarrow$ paraconsistency?

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Proof.

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Since $A \wedge \sim A \to B$ *fails in* $\Pi'_{\mathbf{E}}$*'s model of relevance, it follows that* $\{A, \sim A\} \nvdash^r_{\Pi'_{\mathbf{E}}} B$. □

Corollary (Relevance $\Longrightarrow \vdash^r$-paraconsistency)

If **L** *is a logic for which the Entailment theorem holds, then* $\vdash^r_{\mathbf{L}}$ *is paraconsistent if* **L** *has the variable sharing property.*

Relevance $\not\Longrightarrow \vdash^h$-paraconsistency

Relevance $\Longrightarrow$ paraconsistency?

Theorem ($\vdash^r_{\Pi'_{\mathbf{E}}}$ is paraconsistent)

Proof.

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Corollary (Relevance $\Longrightarrow$ $\vdash^r$-paraconsistency)

If $\mathbf{L}$ *is a logic for which the Entailment theorem holds, then* $\vdash^r_{\mathbf{L}}$ *is paraconsistent if* $\mathbf{L}$ *has the variable sharing property.*

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Summing up

We have seen that $\Pi'_{\mathbf{E}}$ has the following properties

- Variable sharing
- Entailment theorem
- Enthymematical deduction theorem
- **S4** modality
- Explosive $\vdash^h$
- Mildly paraconsistent $\vdash^r$

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we do hold that the inference from $\overline{A}$ and $A \vee B$ to B is in error: it is a simple inferential mistake, such as only a dog would make [...]. Such an inference commits nothing less than a fallacy of relevance. ([AB75, p. 165])

Moral

Claiming that the inference from A and $\sim A \vee B$ to B is a violation of relevance is generally too strong.

- Claiming that B is $\vdash^r$-inferable from A together with $\sim A \vee B$ does commit one to a fallacy of relevance when A and B share no propositional parameter.
- Claiming that B is $\vdash^h$-inferable from A together with $\sim A \vee B$, however, does not.
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Not good enough!

- Expressing entailment:
  - Because of the Entailment theorem, $\rightarrow$ expresses $\vdash^r$-entailment.
  - Because of the Material Deduction Theorem, $\supset$ expresses $\vdash^h$-entailment.
  - However, since $\supset$ is non-modal, it doesn't capture the full meaning of $\vdash^h$ - namely that *any* world/point ought to be closed under $\vdash^h$ if it validates all of logic.
  - $\Pi'_{\mathbf{E}}$ seems to have no definable conditional which can uniformly express this modal aspect of $\vdash^h$-entailment.
- $\Pi'_{\mathbf{E}}$ does not validate all of **E**, and (Ax16) can't be added to **R** with the Church constant without loosing the variable sharing property: $((A \wedge ({\sim}A \vee \bot) \wedge (\bot \rightarrow \bot))) \rightarrow \bot$ yields $(A \wedge {\sim}A) \rightarrow B)$
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Basic idea:

Add the Ackermann constant $\mathbf{t}$ to $\mathbf{E}$ and $\mathbf{R}$ to get $\mathbf{E^t}$ and $\mathbf{R^t}$, where the defining axioms for $\mathbf{t}$ are $\mathbf{t} \to (A \to A)$ and $(\mathbf{t} \to A) \to A$.

Definition

Æ and $\mathbf{M}$ are defined as, respectively, $\mathbf{E^t}$ and $\mathbf{R^t}$ plus the axiom $\mathbf{t} \wedge \mathbf{f} \to A$, where $\mathbf{f} =_{df} {\sim}\mathbf{t}$.

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Results for **Æ** and **M**

- $\vdash^h$ is explosive for both **Æ** and **M**
- $\vdash^r$ is mildly paraconsistent for both **Æ** and **M**
- **M** is a conservative extension of **R** (corollary from Meyer's result on **R** with Boolean negation)
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- **Æ** and **M** satisfy the variable sharing property suitably extended to cover $\mathbf{t}$
- **Æ** and **M** validates the Entailment Theorem, so $\rightarrow$ expresses $\vdash^r$-entailment
- **Æ** and **M** have the enthymematical deduction theorem with $\mathbf{t}$ replacing the conjunction of axioms:

$$\Theta \vdash^h A \Longleftrightarrow \emptyset \vdash^h \bigwedge_{i \leq n} \theta_i \mapsto A$$

  $A \mapsto B =_{df} A \wedge \mathbf{t} \rightarrow B$ therefore *expresses* $\vdash^h$-entailment.

- $$\Box(A \supset B) \dashv\vdash^h A \mapsto B \qquad\qquad {\sim}\Box(A \supset B) \dashv\vdash^h {\sim}(A \mapsto B)$$

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(a) Restricted language:
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Conclusions

Relevance; Disjunctive Syllogism; Intra-Theoretical Pluralism

- The concept of premise-use often appealed to by relevantists does yield a special consequence relation, but is not *sufficient* for relevance.
- A&B's relevant consequence relation is simply a way to generate a new consequence relation from an old one and has nothing special to do with relevance.
- The heart of *relevance* is at best variable sharing together with the claim that the conditional for which variable sharing holds *expresses* a consequence relation.
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