Dialetheic Paths to Triviality

Tore Fjetland Øgaard

Goal

- Show triviality & non-conservativeness of naïve theories in some relevant logics
- Give an overview over what relevant logics might treat such theories non-trivially.

Permutation

- The first batch of proofs use some form of permutation:

$$
\begin{array}{ll}
\text{(Ax11)} & (A \to (B \to C)) \to (B \to (A \to C)) \\
\text{(R9)} & A \to (B \to C) \vdash B \to (A \to C) \\
(\delta^{+}) & A \to (B \to C), B \vdash B \to (A \to C) \\
(\delta^{*}) & A \to (\top \to C) \vdash \top \to (A \to C) \\
(\delta) & A \to (B \to C), B \vdash A \to C
\end{array}
$$

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(\delta^*) & A \to (\top \to C) \vdash \top \to (A \to C) \\
(\delta) & A \to (B \to C), B \vdash A \to C
\end{array}
$$

"Various natural arguments require the use of principles that involve nested $\to$s, such as Permutation, $\{\alpha \to (\beta \to \gamma)\} \vdash \beta \to (\alpha \to \gamma)$. [...] Whether it can be added while maintaining non-triviality is not known."

(Graham Priest: *In Contradiction* (2006)/"Paraconsistent Set Theory" (2011))

Structure of talk

- First task: Present the logics

- Second task: Triviality & non-conservativeness proofs
  (i) Proofs Part I: Proofs using permutation & excluded middle
  (ii) Proofs Part II: Proofs using conjunctive syllogism

- If time permits:
  (iii) Proofs Part III: Proofs using fusion:

$$\text{(Residuation)} \quad (A \circ B) \to C \dashv\vdash A \to (B \to C)$$

Main theorem: Triviality given reasoning by cases & Ackermann constant **t** & contraposition axiom & excluded middle.

Logics: BB

Ax1	$A \to A$	
Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$	
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$	
Ax4	$\neg\neg A \to A$	
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$	
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom
Ax12	$((A \to B) \wedge (B \to C)) \to (A \to C)$	conjunctive syllogism
Ax13	$A \vee \neg A$	excluded middle
Ax14	$(A \to \neg A) \to \neg A$	consequentia mirabilis
Ax15	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
R1	$A, B \vdash A \wedge B$	adjunction
R2	$A, A \to B \vdash B$	modus ponens
R3	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule
R4	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule
R5	$A \to \neg B \vdash B \to \neg A$	contraposition rule
R6	$A \to B, A \to C \vdash A \to (B \wedge C)$	lattice conjunction
R7	$A \to C, B \to C \vdash (A \vee B) \to C$	lattice disjunction
R8	$A \to (B \to C), B \vdash A \to C$	δ
R9	$A \to (B \to C) \vdash B \to (A \to C)$	permutation rule

BB = Ax1-Ax5, R1-R7

B = **BB** +Ax6 +Ax7; -R6 -R7

DW = **B** +Ax8; -R5

TW = **DW** +Ax9 +Ax10; -R3 -R4

EW = **TW** +R8

RW = **TW** +Ax11

R = **RW** +Ax12
= **RW** +Ax14
= **RW** +Ax15

Logics: B

Ax1	$A \to A$		
Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$		
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$		
Ax4	$\neg\neg A \to A$		
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$		
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$	**BB** $=$ Ax1-Ax5, R1-R7
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$	
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom	**B** $=$ **BB** $+$Ax6 $+$Ax7; -R6 -R7
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom	
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom	**DW** $=$ **B** $+$Ax8; -R5
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom	
Ax12	$((A \to B) \wedge (B \to C)) \to (A \to C)$	conjunctive syllogism	**TW** $=$ **DW** $+$Ax9 $+$Ax10; -R3 -R4
Ax13	$A \vee \neg A$	excluded middle	
Ax14	$(A \to \neg A) \to \neg A$	consequentia mirabilis	**EW** $=$ **TW** $+$R8
Ax15	$(A \to (A \to B)) \to (A \to B)$	contraction axiom	
R1	$A, B \vdash A \wedge B$	adjunction	**RW** $=$ **TW** $+$Ax11
R2	$A, A \to B \vdash B$	modus ponens	
R3	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule	
R4	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule	**R** $=$ **RW** $+$Ax12
R5	$A \to \neg B \vdash B \to \neg A$	contraposition rule	$=$ **RW** $+$Ax14
R6	$A \to B, A \to C \vdash A \to (B \wedge C)$	lattice conjunction	$=$ **RW** $+$Ax15
R7	$A \to C, B \to C \vdash (A \vee B) \to C$	lattice disjunction	
R8	$A \to (B \to C), B \vdash A \to C$	δ	
R9	$A \to (B \to C) \vdash B \to (A \to C)$	permutation rule	

Logics: DW

Ax1	$A \to A$			
Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$			
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$			
Ax4	$\neg\neg A \to A$			
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$			
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$		**BB** $=$ Ax1-Ax5, R1-R7
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$		
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom		**B** $=$ **BB** $+$Ax6 $+$Ax7; -R6 -R7
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom		
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom		**DW** $=$ **B** $+$Ax8; -R5
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom		
Ax12	$((A \to B) \wedge (B \to C)) \to (A \to C)$	conjunctive syllogism		**TW** $=$ **DW** $+$Ax9 $+$Ax10; -R3 -R4
Ax13	$A \vee \neg A$	excluded middle		
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Ax15	$(A \to (A \to B)) \to (A \to B)$	contraction axiom		
R1	$A, B \vdash A \wedge B$	adjunction		**RW** $=$ **TW** $+$Ax11
R2	$A, A \to B \vdash B$	modus ponens		
R3	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule		
R4	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule		**R** $=$ **RW** $+$Ax12
R5	$A \to \neg B \vdash B \to \neg A$	contraposition rule		$=$ **RW** $+$Ax14
R6	$A \to B, A \to C \vdash A \to (B \wedge C)$	lattice conjunction		$=$ **RW** $+$Ax15
R7	$A \to C, B \to C \vdash (A \vee B) \to C$	lattice disjunction		
R8	$A \to (B \to C), B \vdash A \to C$	δ		
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Logics: TW

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Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$
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Logics: EW

Ax1	$A \to A$		
Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$		
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$		
Ax4	$\neg\neg A \to A$		
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$		
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$	
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$	
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom	
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom	
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom	
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom	
Ax12	$((A \to B) \wedge (B \to C)) \to (A \to C)$	conjunctive syllogism	
Ax13	$A \vee \neg A$	excluded middle	
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Logics: RW

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Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$		
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$		
Ax4	$\neg\neg A \to A$		
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$		
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$	
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$	
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom	
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Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom	
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Logics: R

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Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$	
Ax4	$\neg\neg A \to A$	
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$	
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$
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Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom
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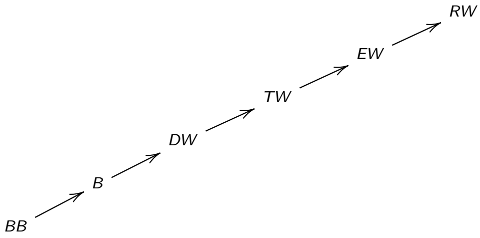
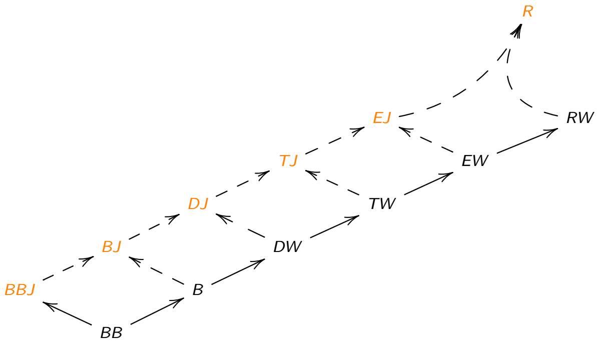
Basic contraction-free relevant logics



J-logics

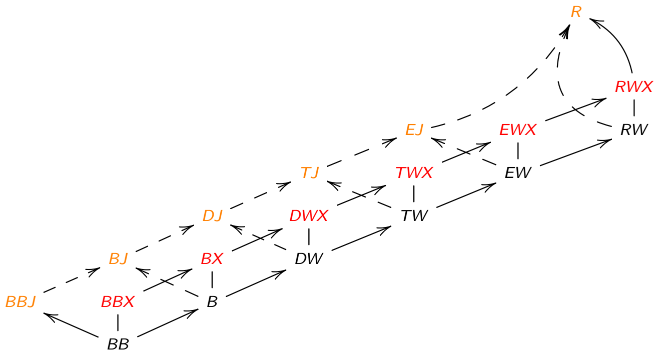
$\mathsf{J} = +\text{Ax12: } ((A \to B) \wedge (B \to C)) \to (A \to C)$



DJ: Ross Brady - *Universal Logic* (2006)

X-logics

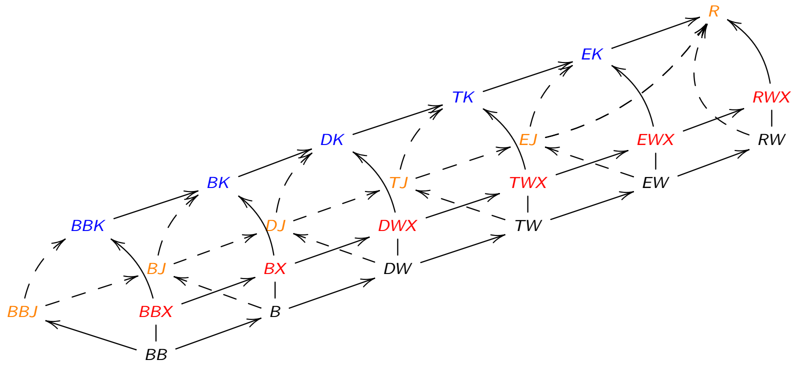
J = +Ax12: $((A \to B) \wedge (B \to C)) \to (A \to C)$
X = +Ax13: $A \vee \neg A$



BX: JC Beall - *Spandrels of Truth* (2009)

K-logics

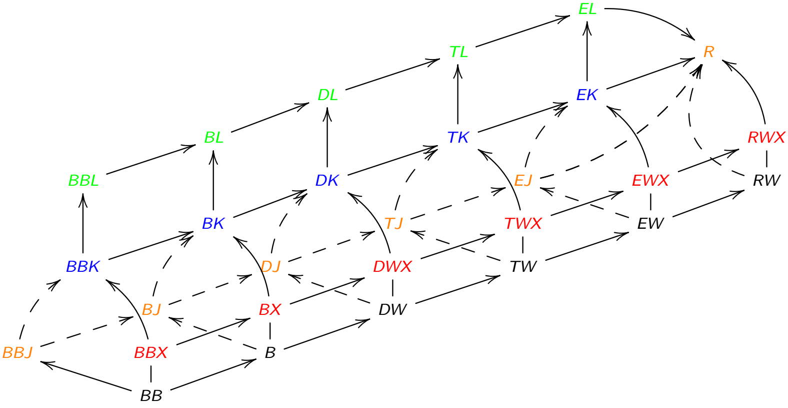
J = +Ax12: $((A \to B) \land (B \to C)) \to (A \to C)$
X = +Ax13: $A \lor \neg A$
K = +Ax12 +Ax13



DK: Richard Routley - "Ultralogic as universal" (1977)
TK: Zach Weber - "Notes on Inconsistent Set Theory" (2013)

L-logics

J = +Ax12: $((A \to B) \land (B \to C)) \to (A \to C)$
X = +Ax13: $A \lor \neg A$
K = +Ax12 +Ax13
L = +Ax12 +Ax14: $(A \to \neg A) \to \neg A$



DL: Zach Weber - "Extensionality and Restriction in Naive Set Theory" (2010)
TL: Zach Weber - "Transfinite Numbers in Paraconsistent Set Theory" (2010)

Some rules and equivalent principles in **BB**

$$\begin{aligned}
&(\text{transitivity}) \quad A \to B, B \to C \vdash_{\mathbf{BB}} A \to C \\
&(\text{leftER}) \quad A \to (B \to C), D \to B \vdash_{\mathbf{BB}} A \to (D \to C) \\
&(\text{rightER}) \quad A \to (B \to C), C \to D \vdash_{\mathbf{BB}} A \to (B \to D)
\end{aligned}$$

- Interderivable in **BB**:

$$\begin{array}{ll}
(\text{Ax13}) & A \vee \neg A \\
(\to\text{-ExMid}) & A \to \neg A \vdash \neg A
\end{array}$$

Other Principles

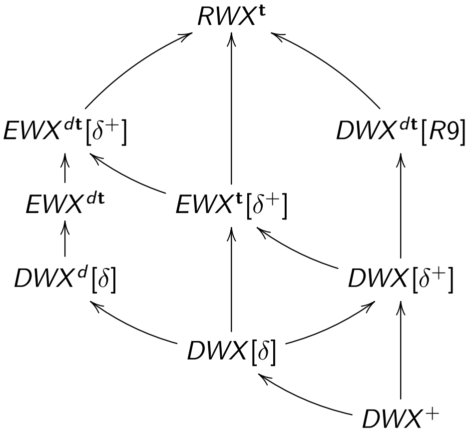
- Reasoning by cases

$$(\text{M1}) \quad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

- Ackermann constant:

$$(\mathbf{t}\text{-rule}) \quad A \dashv\vdash \mathbf{t} \to A.$$

More logics - naming conventions



(δ) $A \to (B \to C), B \vdash A \to C$
(δ^+) $A \to (B \to C), B \vdash B \to (A \to C)$
(R9) $A \to (B \to C) \vdash B \to (A \to C)$

Naïve Theories

Naïve truth theory.

$$ⓝ \vdash A \leftrightarrow T\ulcorner A \urcorner$$

Naïve Theories

Naïve truth theory.

$$ⓝ \vdash A \leftrightarrow T\ulcorner A \urcorner$$

$$\begin{aligned}
C &\leftrightarrow (C \to \bot) \\
C &\leftrightarrow (\top \to \neg C) \\
C &\leftrightarrow (\top \to (C \to \bot)) \\
C &\leftrightarrow (\neg(C \to \bot) \to \bot)
\end{aligned}$$

$$\begin{aligned}
C &\leftrightarrow ((C \circ C) \to \bot) \\
C &\leftrightarrow (\neg((C \circ \mathbf{t}) \to \bot) \to \bot) \\
C &\leftrightarrow (\top \to \neg(C \circ \top)) \\
C &\leftrightarrow ((\top \circ C) \to \bot)
\end{aligned}$$

$\top$ and $\bot$ are definable

Church constants:

$$\bot =_{df} \forall x T(x)$$
$$\top =_{df} \exists x T(x)$$

Easy theorems:

$$ⓝ \vdash \bot \to A$$
$$ⓝ \vdash A \to \top$$

Proofs part I

To be shown:

ⓝ $\vdash_{\mathbf{BBX}\overset{\neg}{\rightarrow}[R9]} \bot$

ⓝ $\vdash_{\mathbf{DWX}[\delta^{+}]} \bot$

ⓝ $\vdash_{\mathbf{BBX}^{d}[\delta]} (\neg(A \rightarrow A) \rightarrow A) \rightarrow A$

ⓝ $\vdash_{\mathbf{BBX}^{d}[\delta^{+}]} \bot$ (corollary)

RWX is not Curry Paraconsistent

- Slaney showed that **RWX** validates too much contraction:

$$\vdash_{\mathbf{RWX}} (A \to (A \to \bot)) \to (A \to \bot)$$

Slaney's theorem:

$$ⓝ \vdash_{\mathbf{RWX}} \bot$$

RWX is not Curry Paraconsistent

- Slaney showed that **RWX** validates too much contraction:

$$\vdash_{\mathbf{RWX}} (A \to (A \to \bot)) \to (A \to \bot)$$

Slaney's theorem:

$$ⓝ \vdash_{\mathbf{RWX}} \bot$$

- Not evident where to "pin the blaim".

  Besides excluded middle, the three most salient principles used in the proof are:

Ax7	$((A \to C) \land (B \to C)) \to ((A \lor B) \to C)$	strong lattice $\lor$
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom

Blaim Permutation I

Theorem

ⓝ $\vdash_{\mathbf{BBX}\vec{\neg}[R9]} \bot$

Proof.

Blaim Permutation I

Theorem

ⓝ $\vdash_{\mathbf{BBX}\vec{\neg}[R9]} \bot$

Proof.

(1)	$C \leftrightarrow (\neg(C \to \bot) \to \bot)$	*naïve theory*
(2)	$(\neg(C \to \bot) \to \bot) \to (\neg(C \to \bot) \to \bot)$	*Ax1*
(3)	$\neg(C \to \bot) \to ((\neg(C \to \bot) \to \bot) \to \bot)$	*2, R9*
(4)	$\neg(C \to \bot) \to (C \to \bot)$	*1, 3, leftER*
(5)	$C \to \bot$	*4, $\to$-ExMid*
(6)	$C \to (\top \to \bot)$	*5 + def. of $\bot$*
(7)	$\top \to (C \to \bot)$	*6, δ^**
(8)	$\neg(C \to \bot) \to \bot$	*7, R5*
(9)	C	*1, 8, R2*
(10)	$\bot$	*5, 9, R2*

□

How about δ^+?

$$
\begin{array}{ll}
(\mathrm{Ax7}) & ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C) \\
(\delta) & A \to (B \to C), B \vdash A \to C \\
(\delta^*) & A \to (\top \to C) \vdash \top \to (A \to C) \\
(\delta^+) & A \to (B \to C), B \vdash B \to (A \to C)
\end{array}
$$

Lemma

$\vdash_{\mathbf{BBX}[Ax7,\delta]} ((A \to C) \wedge (\neg A \to C)) \to C$

Proof.

How about δ^+?

$$
\begin{array}{ll}
\text{(Ax7)} & ((A \to C) \land (B \to C)) \to ((A \lor B) \to C) \\
(\delta) & A \to (B \to C), B \vdash A \to C \\
(\delta^*) & A \to (\top \to C) \vdash \top \to (A \to C) \\
(\delta^+) & A \to (B \to C), B \vdash B \to (A \to C)
\end{array}
$$

Lemma

$\vdash_{\mathbf{BBX}[Ax7,\delta]} ((A \to C) \land (\neg A \to C)) \to C$

Proof.

$$
\begin{array}{lll}
(1) & ((A \to C) \land (\neg A \to C)) \to ((A \lor \neg A) \to C) & \mathit{Ax7} \\
(2) & A \lor \neg A & \mathit{Ax13} \\
(3) & ((A \to C) \land (\neg A \to C)) \to C & \mathit{1,\ 2,\ \delta}
\end{array}
$$

□

Blaim Permutation II: ⓝ $\vdash_{\mathbf{DWX}[\delta^+]} \bot$

Blaim Permutation II: ⓝ $\vdash_{\mathbf{DWX}[\delta^+]} \bot$

(1)	$C \leftrightarrow (\top \to (C \to \bot))$	naïve theory
(2)	$[((C \to \bot) \to \neg C) \wedge (\neg(C \to \bot) \to \neg C)] \to \neg C$	lemma
(3)	$(C \to \bot) \to (\top \to \neg C)$	Ax8
(4)	$\top \to ((C \to \bot) \to \neg C)$	3, δ^*
(5)	$\top \to (C \to (C \to \bot))$	1, δ^*
(6)	$\top \to (\neg(C \to \bot) \to \neg C)$	5, Ax8
(7)	$\top \to [((C \to \bot) \to \neg C) \wedge (\neg(C \to \bot) \to \neg C)]$	4, 6, R6
(8)	$\top \to \neg C$	2, 7, transitivity
(9)	$C \to \bot$	8, R5
(10)	$C \to (\top \to \bot)$	9 + def. of $\bot$
(11)	$\top \to (C \to \bot)$	10, δ^*
(12)	C	1, 11, R2
(13)	$\bot$	9, 12, R2

Cut δ!

Theorem

Ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(A \to A) \to A) \to A$

Cut δ!

Theorem

ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(A \to A) \to A) \to A$

(1)	$C \leftrightarrow (\neg(C \to A) \to A)$	*naïve theory*
(2)	$(C \to A) \vee \neg(C \to A)$	*Ax13*
(3)	$\quad \neg(C \to A)$	*assumption for M1*
(4)	$\quad C \to A$	*1, 2, δ*
(5)	$C \to A$	*2, 3-4, M1*
(6)	$(A \to A) \to (C \to A)$	*5, R3*
(7)	$\neg(C \to A) \to \neg(A \to A)$	*6, R5*
(8)	$(\neg(A \to A) \to A) \to (\neg(C \to A) \to A)$	*7, R3*
(9)	$(\neg(A \to A) \to A) \to A$	*1, 5, 8, transitivity*

Cut δ! (cont.)

Corollary

ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(\bot \to \bot) \to \bot) \to \bot$ *Above theorem with* $(A = \bot)$

$\vdash_{\mathbf{BB}[\delta^*]} \neg(\bot \to \bot) \to \bot$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta^+]} \bot$ *(Given M1, one can drop Ax7&Ax8)*

Proofs II - Conjunctive Syllogism & Consequentia Mirabilis

(Ax12)	$((A \to B) \land (B \to C)) \to (A \to C)$	Conjunctive Syllogism
(Ax14)	$(A \to \neg A) \to \neg A$	Consequentia Mirabilis

- To be shown:

(I) ⓝ $\vdash_{\mathbf{BB}[Ax14,\delta^+]} \bot$

(II) $\vdash_{\mathbf{BBJ^+[R9]}} (A \to (A \to B)) \to (A \to B)$

(III) ⓝ $\vdash_{\mathbf{EJ^+}} \bot$

(IV) ⓝ $\vdash_{\mathbf{TL}} (A \to (\neg A \to \neg A)) \to \neg A$

I: δ^+ & Ax14

Theorem

ⓝ$\vdash_{\mathbf{BB}[Ax14,\delta^+]} \perp$

Proof.

I: δ^+ & Ax14

Theorem

ⓝ$\vdash_{\mathbf{BB}[Ax14,\delta^+]} \bot$

Proof.

(1)	$C \leftrightarrow (\top \rightarrow \neg C)$	*naïve theory*
(2)	$\top \rightarrow (C \rightarrow \neg C)$	*1, δ^**
(3)	$(C \rightarrow \neg C) \rightarrow \neg C$	*Ax14*
(4)	$\top \rightarrow \neg C$	*2, 3, transitivity*
(5)	C	*1, 4, R2*
(6)	$\bot$	*4, 5, modus tollens*

□

II: Deriving contraction

$\vdash_{\mathbf{BB}^{+}[\mathbf{R9}]} A \to ((A \to B) \to B)$ (assertion)

Theorem

$\vdash_{\mathbf{BBJ}^{+}[\mathbf{R9}]} (A \to (A \to B)) \to (A \to B)$

Proof.

II: Deriving contraction

$\vdash_{\mathbf{BB}^{+}[\mathbf{R9}]} A \to ((A \to B) \to B)$ (assertion)

Theorem

$\vdash_{\mathbf{BBJ}^{+}[\mathbf{R9}]} (A \to (A \to B)) \to (A \to B)$

Proof.

(1)	$A \to ((A \to B) \to B)$	*assertion*
(2)	$A \to ((A \to (A \to B)) \to (A \to B))$	*assertion*
(3)	$A \to [\{(A \to (A \to B)) \to (A \to B)\} \wedge \{(A \to B) \to B\}]$	*1, 2, R6*
(4)	$[\{(A \to (A \to B)) \to (A \to B)\} \wedge \{(A \to B) \to B\}] \to ((A \to (A \to B)) \to B)$	*Ax12*
(5)	$A \to ((A \to (A \to B)) \to B)$	*3, 4, trans.*
(6)	$(A \to (A \to B)) \to (A \to B)$	*5, R9*

□

TJ, EJ = TJ + δ, TL = TJ + Ax14

Ax1	$A \to A$	
Ax2	$A \to (A \vee B)$ and $B \to (A \vee B)$	
Ax3	$(A \wedge B) \to A$ and $(A \wedge B) \to B$	
Ax4	$\neg\neg A \to A$	
Ax5	$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$	
Ax6	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	strong lattice $\wedge$
Ax7	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$	strong lattice $\vee$
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
Ax12	$((A \to B) \wedge (B \to C)) \to (A \to C)$	conjunctive syllogism
Ax14	$(A \to \neg A) \to \neg A$	consequentia mirabilis
R1	$A, B \vdash A \wedge B$	adjunction
R2	$A, A \to B \vdash B$	modus ponens
R8	$A \to (B \to C), B \vdash A \to C$	δ

TJ lemma

Lemma

For any sentence A, there is a sentence C such that

$$ⓝ \vdash_{\mathbf{BB}^{+}[\mathbf{Ax9},\mathbf{Ax12}]} C \to ((A \to A) \to A)$$

Proof.

TJ lemma

Lemma

For any sentence A, there is a sentence C such that

$$ⓝ \vdash_{\mathbf{BB^+[Ax9,Ax12]}} C \to ((A \to A) \to A)$$

Proof.

(1)	$C \leftrightarrow (C \to A)$	*naïve theory*
(2)	$(C \to A) \to ((A \to A) \to (C \to A))$	*Ax9*
(3)	$C \to ((A \to A) \to C)$	*1, leftER+trans.*
(4)	$C \to [((A \to A) \to C) \wedge (C \to A)]$	*1, 3, R6*
(5)	$[((A \to A) \to C) \wedge (C \to A)] \to ((A \to A) \to A$	*Ax12*
(6)	$C \to ((A \to A) \to A)$	*4, 5, trans.*

□

III: EJ^{+} is too strong!

Theorem

ⓝ $\vdash_{\mathbf{BB}^{+}[\mathbf{Ax9},\mathbf{Ax12},\delta]} \bot$

Proof.

III: EJ^+ is too strong!

Theorem

ⓝ $\vdash_{\mathbf{BB}^+[\mathbf{Ax9},\mathbf{Ax12},\delta]} \bot$

Proof.

(1)	$C \leftrightarrow (C \to \bot)$	*naïve theory*
(2)	$C \to ((\bot \to \bot) \to \bot)$	*TJ-lemma*
(3)	$\bot \to \bot$	*Ax1*
(4)	$C \to \bot$	*2, 3, δ*
(5)	C	*1, 4, R2*
(6)	$\bot$	*4, 5, R2*

□

IV: TL is non-conservative

Theorem

ⓝ $\vdash_{\mathbf{BB[Ax9,Ax12,Ax14]}} (A \rightarrow (\neg A \rightarrow \neg A)) \rightarrow A$

IV: TL is non-conservative

Theorem

ⓝ $\vdash_{\mathbf{BB[Ax9,Ax12,Ax14]}} (A \to (\neg A \to \neg A)) \to A$

(1)	$C \leftrightarrow (C \to \neg A)$	*naïve theory*
(2)	$C \to ((\neg A \to \neg A) \to \neg A)$	*TJ-lemma*
(3)	$(A \to (\neg A \to \neg A)) \to [((\neg A \to \neg A) \to \neg A) \to (A \to \neg A)]$	*Ax9*
(4)	$(A \to (\neg A \to \neg A)) \to (C \to (A \to \neg A))$	*2, 3, leftER*
(5)	$(A \to \neg A) \to \neg A$	*Ax14*
(6)	$(A \to (\neg A \to \neg A)) \to (C \to \neg A)$	*4, 5, rightER*
(7)	$(A \to (\neg A \to \neg A)) \to C$	*1, 5, trans.*
(8)	$(A \to (\neg A \to \neg A)) \to ((\neg A \to \neg A) \to \neg A)$	*2, 7, trans.*
(9)	$(A \to (\neg A \to \neg A)) \to$ $[(A \to (\neg A \to \neg A)) \wedge ((\neg A \to \neg A) \to \neg A)]$	*8, R6*
(10)	$[(A \to (\neg A \to \neg A)) \wedge ((\neg A \to \neg A) \to \neg A)] \to (A \to \neg A)$	*Ax12*
(11)	$(A \to (\neg A \to \neg A)) \to (A \to \neg A)$	*8, 9, trans.*
(12)	$(A \to (\neg A \to \neg A)) \to \neg A$	*5, 11, trans.*

Possible logics for ⓝ

ⓝ $\vdash_{\mathbf{BBX}\overset{\neg}{\rightarrow}[R9]} \bot$

ⓝ $\vdash_{\mathbf{DWX}[\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(A \rightarrow A) \rightarrow A) \rightarrow A$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta^+]} \bot$

Possible logics for ⓝ

ⓝ $\vdash_{\mathbf{BBX}^{\overset{\neg}{\rightarrow}}[R9]} \bot$

ⓝ $\vdash_{\mathbf{DWX}[\delta^{+}]} \bot$

ⓝ $\vdash_{\mathbf{BBX}^{d}[\delta]} (\neg(A \to A) \to A) \to A$

ⓝ $\vdash_{\mathbf{BBX}^{d}[\delta^{+}]} \bot$

ⓝ $\vdash_{\mathbf{BB}[Ax14,\delta^{+}]} \bot$

ⓝ $\vdash_{\mathbf{BBJ}^{+}[R9]} \bot$

ⓝ $\vdash_{\mathbf{EJ}^{+}} \bot$

ⓝ $\vdash_{\mathbf{TL}} (A \to (\neg A \to \neg A)) \to \neg A$

Possible logics for ⓝ

ⓝ $\vdash_{\mathbf{BBX}\overset{\neg}{\rightarrow}[R9]} \bot$

ⓝ $\vdash_{\mathbf{DWX}[\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(A \rightarrow A) \rightarrow A) \rightarrow A$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BB}[Ax14,\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BBJ}^+[R9]} \bot$

ⓝ $\vdash_{\mathbf{EJ}^+} \bot$

ⓝ $\vdash_{\mathbf{TL}} (A \rightarrow (\neg A \rightarrow \neg A)) \rightarrow \neg A$

Brady:

ⓝ $\nvdash_{\mathbf{TK}^{dt}} \bot$

ⓝ $\nvdash_{\mathbf{DR}^{dt}} \bot$

$\mathbf{DR}^{dt} = \mathbf{DK}^{dt} + A \vdash \neg(A \rightarrow \neg A)$

Possible logics for ⓝ

ⓝ $\vdash_{\mathbf{BBX}^{\vec{\neg}}[R9]} \bot$

ⓝ $\vdash_{\mathbf{DWX}[\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta]} (\neg(A \to A) \to A) \to A$

ⓝ $\vdash_{\mathbf{BBX}^d[\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BB}[Ax14,\delta^+]} \bot$

ⓝ $\vdash_{\mathbf{BBJ}^+[R9]} \bot$

ⓝ $\vdash_{\mathbf{EJ}^+} \bot$

ⓝ $\vdash_{\mathbf{TL}} (A \to (\neg A \to \neg A)) \to \neg A$

Brady:

ⓝ $\nvdash_{\mathbf{TK}^{dt}} \bot$

ⓝ $\nvdash_{\mathbf{DR}^{dt}} \bot$

$\mathbf{DR}^{dt} = \mathbf{DK}^{dt} + A \vdash \neg(A \to \neg A)$

Unknown:

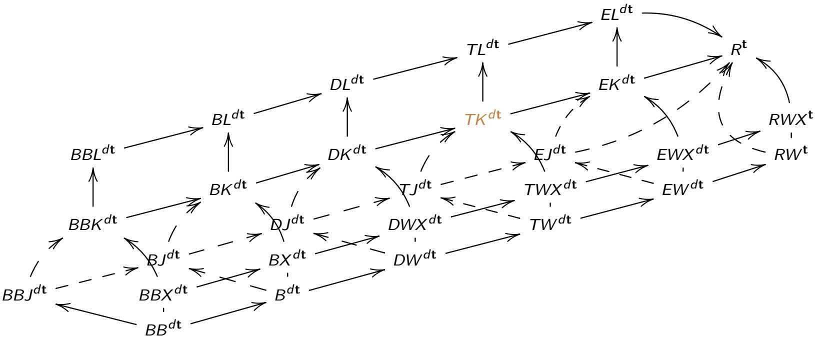
$\mathbf{TW}^{dt}[Ax14]$

$\mathbf{DL}^{dt}$

$\mathbf{DJ}^{dt}[\delta^+]$

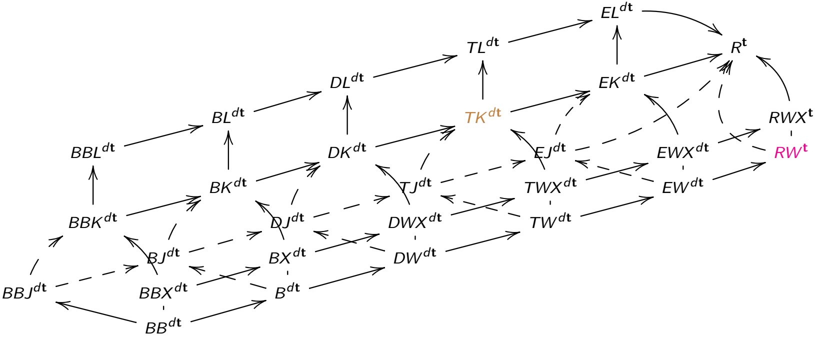
Non-trivial

Non-trivial



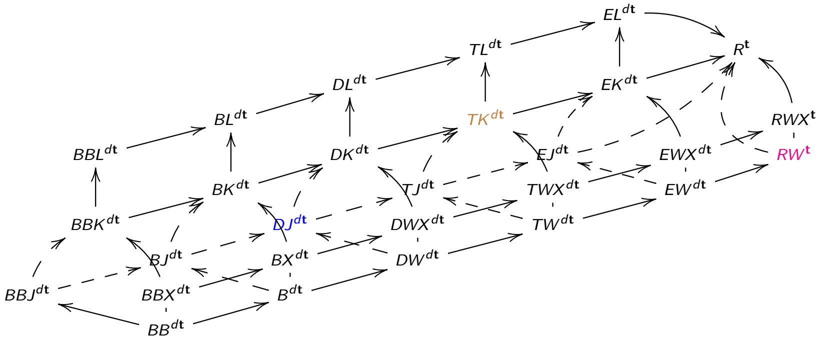
Want Permutation Axiom/Rule?

Non-trivial
Ax11/R9



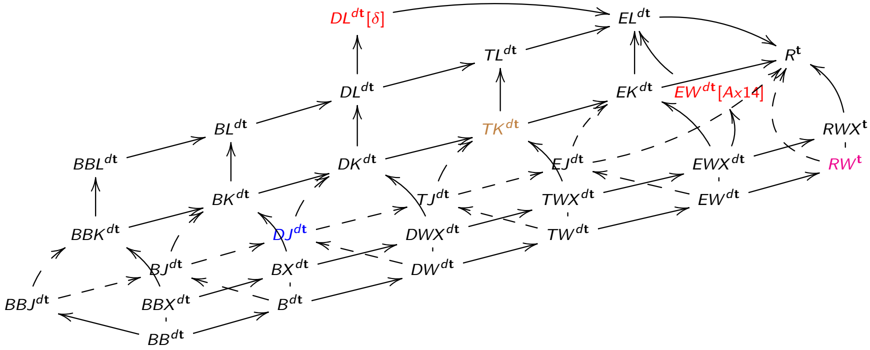
Want δ^+?

Non-trivial
Ax11/R9
δ^+



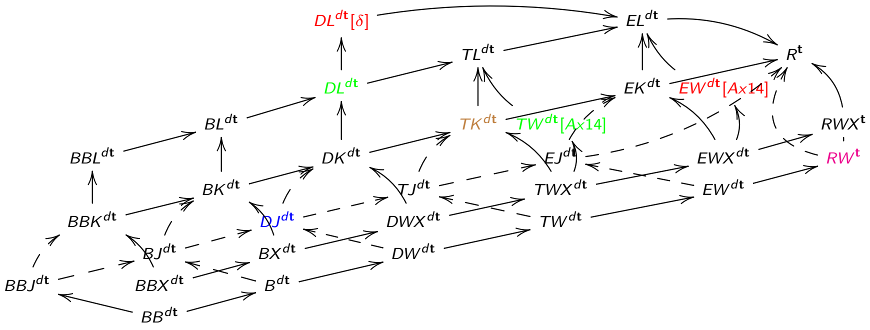
Want δ non-conservatively?

Non-trivial
Ax11/R9
δ^+
δ, non-con



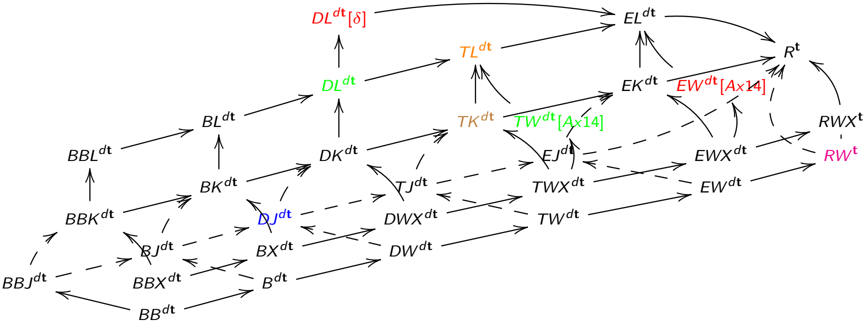
No Permutation, Conservative

Non-trivial
Ax11/R9
δ^+
δ, non-con
No permutation



No Permutation, non-Conservative

Non-trivial
Ax11/R9
δ^+
δ, non-con
No permutation
No permutation, non-conservative



Proofs part III

Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
Ax14	$(A \to \neg A) \to \neg A$	consequentia mirabilis

To be shown:

(I) ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \bot$ (Dunn/Meyer/Slaney)

(II) ⓝ$\vdash_{\mathbf{BBX}^{dt\circ}[Ax8]} \bot$

(III) ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$ (corollary)

(IV) ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \bot$

(V&VI) ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \bot$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \bot$

Fusion I - Dunn/Meyer/Slaney

$$\begin{array}{ll} \text{(Residuation)} & (A \circ B) \to C \dashv\vdash A \to (B \to C) \\ \text{(Ax12)} & ((A \to B) \wedge (B \to C)) \to (A \to C) \end{array}$$

Theorem

ⓝ$\vdash_{\mathbf{BBJ}^{\circ +}} \bot$ *(Dunn/Meyer/Slaney)*

Proof.

Fusion I - Dunn/Meyer/Slaney

(Residuation)	$(A \circ B) \to C \dashv\vdash A \to (B \to C)$
(Ax12)	$((A \to B) \wedge (B \to C)) \to (A \to C)$

Theorem

$ⓝ\vdash_{\mathbf{BBJ}^{\circ+}} \bot$ *(Dunn/Meyer/Slaney)*

Proof.

(1)	$C \leftrightarrow ((C \circ C) \to \bot)$	*naïve theory*
(2)	$(C \circ C) \to (C \circ C)$	*Ax1*
(3)	$C \to (C \to (C \circ C))$	*2, residuation*
(4)	$C \to ((C \to (C \circ C)) \wedge ((C \circ C) \to \bot)$	*1, 3, R6*
(5)	$((C \to (C \circ C)) \wedge ((C \circ C) \to \bot) \to (C \to \bot)$	*Ax12*
(6)	$C \to (C \to \bot)$	*4, 5, transitivity*
(7)	$(C \circ C) \to \bot$	*6, residuation*
(8)	C	*1, 7, R2*
(9)	$\bot$	*6, 8, R2 twice*

□

Fusion II: $\textcircled{n} \vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \bot$

Fusion II: ⓝ$\vdash_{\mathbf{BBX}^{dt\circ}[Ax8]} \bot$

(1)	$C \leftrightarrow (\neg((C \circ \mathbf{t}) \to \bot) \to \bot)$	naïve theory
(2)	$((C \circ \mathbf{t}) \to \bot) \vee \neg((C \circ \mathbf{t}) \to \bot)$	Ax13
(3)	$\quad \neg((C \circ \mathbf{t}) \to \bot)$	assumption for M1
(4)	$\quad \mathbf{t} \to \neg((C \circ \mathbf{t}) \to \bot)$	3, **t**-rule
(5)	$\quad (\neg((C \circ \mathbf{t}) \to \bot) \to \bot) \to (\mathbf{t} \to \bot)$	4, suffixing rule
(6)	$\quad C \to (\mathbf{t} \to \bot)$	1, 5, transitivity
(7)	$\quad (C \circ \mathbf{t}) \to \bot$	6, residuation
(8)	$(C \circ \mathbf{t}) \to \bot$	2, 3-7, M1
(9)	$(\top \circ \top) \to \top$	def. of $\top$
(10)	$\top \to (\top \to \top)$	9, residuation
(11)	$\top \to (\bot \to \bot)$	10, Ax8
(12)	$\top \to ((C \circ \mathbf{t}) \to \bot)$	8, 11, leftER
(13)	$\neg((C \circ \mathbf{t}) \to \bot) \to \bot$	12, R5
(14)	C	1, 13, R2
(15)	$C \to (\mathbf{t} \to \bot)$	8, residuation
(16)	$\mathbf{t}$	Ax1 + **t**-rule
(17)	$\bot$	14, 15, 16, R2

Fusion III: ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

Fusion III: ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

- The only application of Ax8 in the above proof was in order to derive $\top \to (\bot \to \bot)$. This sentence is also derivable given δ and Ax9

$$(Ax9) \quad (A \to B) \to ((B \to C) \to (A \to C))$$

Fusion III: ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

- The only application of Ax8 in the above proof was in order to derive $\top \to (\bot \to \bot)$. This sentence is also derivable given δ and Ax9

$$(Ax9) \quad (A \to B) \to ((B \to C) \to (A \to C))$$

(1)	$(\top \to \bot) \leftrightarrow \bot$	δ + fiddling
(2)	$(\top \to \top) \to ((\top \to \bot) \to (\top \to \bot))$	Ax9
(3)	$(\top \to \top) \to (\bot \to \bot)$	1, 2, left/rightER
(4)	$(\top \circ \top) \to \top$	def. of $\top$
(5)	$\top \to (\top \to \top)$	4, residuation
(6)	$\top \to (\bot \to \bot)$	3, 5, transitivity

Fusion III: ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

- The only application of Ax8 in the above proof was in order to derive $\top \rightarrow (\bot \rightarrow \bot)$. This sentence is also derivable given δ and Ax9

$$(Ax9) \quad (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$

(1)	$(\top \rightarrow \bot) \leftrightarrow \bot$	δ + fiddling
(2)	$(\top \rightarrow \top) \rightarrow ((\top \rightarrow \bot) \rightarrow (\top \rightarrow \bot))$	Ax9
(3)	$(\top \rightarrow \top) \rightarrow (\bot \rightarrow \bot)$	1, 2, left/rightER
(4)	$(\top \circ \top) \rightarrow \top$	def. of $\top$
(5)	$\top \rightarrow (\top \rightarrow \top)$	4, residuation
(6)	$\top \rightarrow (\bot \rightarrow \bot)$	3, 5, transitivity

Corollary

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$ *(since δ, use $\hat{\mathbf{t}}$.)*

$\mathbf{t}$ is definable given δ

Theorem

In ⓝ *over any logic* $\mathbf{L}$ *which extends* $\mathbf{BB}^+[\delta]$*, there is a definable truth-constant* $\hat{\mathbf{t}}$ *such that*

$$ⓝ, A \vdash_{\mathbf{L}} \hat{\mathbf{t}} \to A \quad \textit{and} \quad ⓝ, \hat{\mathbf{t}} \to A \vdash_{\mathbf{L}} A$$

Proof.

$\mathbf{t}$ is definable given δ

Theorem

In ⓝ *over any logic* $\mathbf{L}$ *which extends* $\mathbf{BB}^+[\delta]$, *there is a definable truth-constant* $\hat{\mathbf{t}}$ *such that*

$$ⓝ, A \vdash_{\mathbf{L}} \hat{\mathbf{t}} \to A \quad \textit{and} \quad ⓝ, \hat{\mathbf{t}} \to A \vdash_{\mathbf{L}} A$$

Proof.

(1)	$\hat{\mathbf{t}} \leftrightarrow \forall x((T(x) \wedge T(\hat{\mathbf{t}})) \to T(x))$	*naïve truth theory*
(2)	$\forall x((T(x) \wedge T(\hat{\mathbf{t}})) \to T(x))$	*logical theorem*
(3)	$\hat{\mathbf{t}}$	*1, 2, R2*
(4)	$\hat{\mathbf{t}} \to A$	*assumption*
(5)	A	*3, 4, R2*
(6)	A	*assumption*
(7)	$A \wedge \hat{\mathbf{t}}$	*3, 6, R1*
(8)	$\hat{\mathbf{t}} \to ((A \wedge \hat{\mathbf{t}}) \to A)$	*1, naïve truth theory*
(9)	$\hat{\mathbf{t}} \to A$	*7, 8, δ*

□

Fusion IV:

Theorem

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \perp$

Proof.

Fusion IV:

Theorem

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \perp$

Proof.

(1)	$C \leftrightarrow (\top \to \neg(C \circ \top))$	*naïve theory*
(2)	$(C \circ \top) \to \neg(C \circ \top)$	*1, residuation*
(3)	$\neg(C \circ \top)$	*2, $\to$-ExMid*
(4)	$(C \circ \top) \to (C \circ \top)$	*Ax1*
(5)	$C \to (\top \to (C \circ \top))$	*4, residuation*
(6)	$C \to (\neg(C \circ \top) \to \perp)$	*5, Ax8*
(7)	$C \to \perp$	*3, 6, δ*
(8)	$C \to (\top \to \perp)$	*7, def. of $\perp$*
(9)	$(C \circ \top) \to \perp$	*8, residuation*
(10)	$\top \to \neg(C \circ \top)$	*9, R5*
(11)	C	*1, 10, R2*
(12)	$\perp$	*7, 11, R2*

□

Fusion V&VI:

Lemma

$\vdash_{\mathbf{BB}^\circ[Ax8]} \top \to (\bot \to \bot)$ $\vdash_{\mathbf{BB}^\circ[Ax9,Ax14]} \top \to (\bot \to \bot)$

$\vdash_{\mathbf{BB}[Ax14]} (A \to B) \to (\neg A \vee B)$ *(counter-example axiom)*

Theorem

ⓝ $\vdash_{\mathbf{BB}^\circ[Ax8,Ax14]} \bot$ ⓝ $\vdash_{\mathbf{BB}^\circ[Ax9,Ax14]} \bot$

Fusion V&VI:

Lemma

$\vdash_{\mathbf{BB}^{\circ}[Ax8]} \top \to (\bot \to \bot)$ $\quad$ $\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \top \to (\bot \to \bot)$

$\vdash_{\mathbf{BB}[Ax14]} (A \to B) \to (\neg A \vee B)$ *(counter-example axiom)*

Theorem

ⓝ $\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \bot$ $\quad$ ⓝ $\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \bot$

(1)	$C \leftrightarrow ((\top \circ C) \to \bot)$	*naïve theory*
(2)	$((\top \circ C) \to \bot) \to (\neg(\top \circ C) \vee \bot)$	*Ax14 (counter-example axiom)*
(3)	$C \to \neg(\top \circ C)$	*1, 2, transitivity + fiddling*
(4)	$(\top \circ C) \to \neg C$	*3, R5*
(5)	$\top \to (C \to \neg C)$	*4, residuation*
(6)	$(C \to \neg C) \to \neg C$	*Ax14*
(7)	$\top \to \neg C$	*5, 6, transitivity*
(8)	$C \to \bot$	*7, R5*
(9)	$\top \to (\bot \to \bot)$	*lemma*
(10)	$\top \to (C \to \bot)$	*8, 9, leftER*
(11)	$(\top \circ C) \to \bot$	*10, residuation*
(12)	C	*1, 11, R2*
(13)	$\bot$	*8, 12, R2*

Possible ∘-logics for ⓝ

ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \perp$

Unknown:

Possible ∘-logics for ⓝ

ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \perp$

Unknown:

$\mathbf{B}^{d\mathbf{t}\circ}[Ax14]$

Possible ∘-logics for ⓝ

ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \perp$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \perp$

Unknown:

$\mathbf{B}^{d\mathbf{t}\circ}[Ax14]$

$\mathbf{BX}^{d\mathbf{t}\circ}[Ax9, Ax10, CEr]$

$CEr = A \vdash \neg(A \to \neg A)$

Possible ∘-logics for ⓝ

ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \bot$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \bot$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \bot$

Unknown:

$\mathbf{B}^{d\mathbf{t}\circ}[Ax14]$

$\mathbf{BX}^{d\mathbf{t}\circ}[Ax9, Ax10, CEr]$

$\mathbf{TWX}^{d\circ}[CEr]$ (dodgy)

$$CEr = A \vdash \neg(A \to \neg A)$$

Possible ∘-logics for ⓝ

ⓝ$\vdash_{\mathbf{BBJ}^{\circ+}} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{d\circ}[Ax9,\delta]} \bot$

ⓝ$\vdash_{\mathbf{BBX}^{\circ}[Ax8,\delta]} \bot$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax8,Ax14]} \bot$

ⓝ$\vdash_{\mathbf{BB}^{\circ}[Ax9,Ax14]} \bot$

Unknown:

$\mathbf{B}^{d\mathbf{t}\circ}[Ax14]$

$\mathbf{BX}^{d\mathbf{t}\circ}[Ax9, Ax10, CEr]$

$\mathbf{TWX}^{d\circ}[CEr]$ (dodgy)

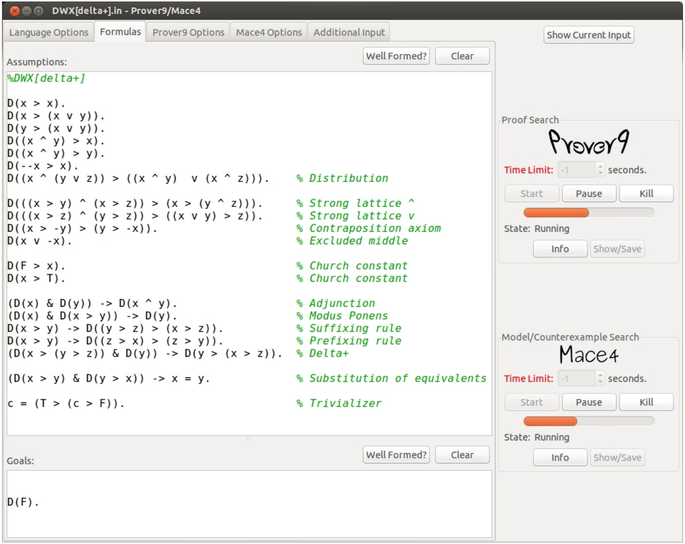
$\mathbf{B}^{d\mathbf{t}\circ}[Ax14, \delta]$ (non-conservative)

$CEr = A \vdash \neg(A \rightarrow \neg A)$

The proofs to be presented have been aided by John Slaney's MaGIC and William McCune's Prover9/Mace4.

```
Terminal
Logic:          DW

Plus:           p v ~p
                p -> (q -> r), q / q -> (p -> r)

Extra:          C <-> (T -> (C -> F))

Fragment:       ->, &, v, ~, t, f, T, F

Definitions:    a <-> b     (a -> b) & (b -> a)
                C        Primitive  (cut)

TTY output:     pretty
File output:    none

Search concludes when 1 matrix found
or when size 14 finished.


   A)xiom     B)adguy      C)onnective    D)elete
   E)xit      F)ragment    G)enerate      H)elp
   I)O        J)ump        K)ill          L)ogic
   M)aGIC     N)o. Procs   O)rder         P)rint Opts
   Q)uit      R)ead        S)tore                        g

Searching.....
```

The proofs to be presented have been aided by John Slaney's MaGIC and William McCune's Prover9/Mace4.



Thank you!